Theoretical frameworks of reference

Introduction
TELMA integrates teams of researchers working on the application of ICT tools for improving mathematics education at school level. The teams are involved both in using and studying a range of existing technological tools and in the design and implementation of new educational tools. The developed researches have been brought forward by the teams referring to a variety of different theoretical frameworks.

The aim of this chapter is to analyse the different theoretical frameworks adopted by TELMA teams, and to compare them individuating some common reading keys. Thus in the following, rather then reporting exhaustively on TELMA researches, we will focus only on those aspects of the researches that contributed to the definition of such reading keys. The keys will be used as lenses through which we will analyse and compare the theoretical frameworks adopted by the teams, and how they contributed to the different researches.

We observe that research on ICT-based teaching and learning, has undergone a deep transformation and such evolution has been caused also by a parallel evolution of the pedagogical and cognitive science theories of reference. As a consequence, the work of TELMA teams is to be intended as a dynamic process undergoing a continuous evolution.

The two main activities characterising the researches of TELMA teams concerning ICT tools, are:

- definition, development and experimental analysis of computer software in the context of mathematical teaching and learning;
- design, realization and analysis of new software in the context of mathematics education.

In the first section of this chapter we will provide some reading keys allowing us to analyse the team’s researches with respect to the activities we described above. The two kinds of activities are often interlaced in the team’s work, in fact the design and development of a new software is often followed by experimentations with pupils and/or teachers. Moreover, in some cases, the experimental phase is a constituting part of the design process (see chapter concerning research methodologies of TELMA teams).

The ICT tools developed and/or used by TELMA teams can be classified according to the following distinction:

- general purpose software, professional software, and software programmable software to be used to develop specific applications.
- Software developed for educational aims, and in particular, for mathematics education.

General purpose, or professional software, such as spreadsheets or CAS (Computer Algebra Systems), are not developed by TELMA teams, but they are analysed in specific experimental settings in order to evaluate their educational potentialities. In the case of educational software, it can be either developed by TELMA teams, or by other producers, like in the case of commercial software (ex. Cabri-Géomètre or Geometer Sketch Pad). A report on the used tools can be found in a dedicated chapter of this book.

In the second section we present the theoretical frameworks elaborated by the TELMA teams to support their researches: an overview allows us to introduce the researches of the teams (area of research, role of technology in the research, etc.); then we present the aims of the researches and finally the use of the theoretical frameworks of reference in the research of each team. Of course this work doesn’t aim at judging or criticize any theoretical framework with respect to others, on the contrary, we assumed the perspective that the value of any theory is not “whether the theory or
framework provides an objective representation of reality” (Badram, 1998), but rather how well a theory can shape an object of study, highlighting relevant issues.

In the third section we present some of the research issues of common interest for TELMA teams, indicating how they approached such issues referring to different theoretical frameworks.

In Annexes of this chapter we collected some key concepts of the educational theories used as a reference framework from the various team and, where possible, we highlighted the role played by ICT tools in such theoretical framework.

The nature of employed technology

The variety of researches of the TELMA teams, suggests to take into account two main kinds of ICT: those (e.g. Aplusix, l’Algebrista, ARI-LAB-2) which have been realized for explicit educational purposes (which we may call educational ICT), and those (e.g. CAS and Spreadsheet) that have been realized for professional purposes (professional ICT). The TELMA teams researches, involving educational ICT, in some cases focus only on the use of ICT in educational practices, in other cases they consider the whole lifecycle of the tools, from the design to the actual use in educational practices and evaluation. In the case of professional ICT, TELMA teams have been focusing only on the educational use of the software, but not in their development.

For instance, the UDE team focuses its research on the development of collaborative and educational components, which are then experimented within small group scenarios. Similarly the team Did@TIC developed and experimented the educational ICT Aplusix, while the team ITD-CNR developed and experimented ARI-LAB-2. The Pisa team on the one hand developed and experimented L’Algebrista, and on the other hand experimented the educational ICT Cabri Géomètre (which was not developed by such team). Also the NKUA team is involved in both the use and development of educational ICT tools in the sense that they assembly kits of microworlds to be used in educational practices, and they experimentation lead to the productions of new environments. Finally, differently from the previous examples, the DIDIREM team focuses on the educational uses of professional ICT such as Computer Algebra Systems (CAS).

Didactical functionalities of ICT tools

Given an ICT tool it is possible to identify its didactical functionalities:

With didactical functionalities we mean those properties (or characteristics) of a given ICT, and (or their) modalities of employment, which may favor or enhance teaching/learning processes according to a specific educational aim.

The aim of this definition is to characterize ICT used for educational purposes and ICT used for other purposes, in the sense that in the latter case no didactical functionalities are taken into account.

The three key elements of the definitions of the didactical functionalities of an ICT tool are then:

1. a set of features/characteristics of the tool
2. an educational aim
3. a modalities of employing the tool in a teaching/learning process referred to the chosen educational aim

For what concerns the features and characteristics of ICT tools, we focus on the distinction between professional and educational ICT. An educational ICT tool provides, because of its nature, a set of such functionalities. In fact we assume that the producers of the tool, not only design it with respect to a set of specific educational goals, but we assume that they also consider the possible modalities of employment of the tools in order to achieve such goals. In other words educational ICT tools are designed together with a set of didactical functionalities.
On the other hand professional ICT tools are not designed considering a possible educational goal and related modalities of employment. In other words they are designed without a set of didactical functionalities. Nevertheless professional ICT tools may provide features that can be interpreted in terms of didactical functionalities, that is, we can identify modalities of employment of such tools aiming at the achievement of a given educational goal.

In general the didactical functionalities can be defined/individuated either at the level of the design phase, or at the educational use phase. Thus in the case of professional ICT, the definition of didactical functionalities occurs only in the utilization phase, whilst in the case of educational ICT, it surely occurs in the first phase, but may occur also in the latter.

In the perspective we are proposing, in order to exploit a given ICT tool as a mean for achieving a given educational goal, it is needed to define the modalities of employment of the tool. In the variety of the researches involving TELMA teams, we found a common effort to describe how the used tools are employed as means for achieving educational goals. Such descriptions depend strictly on each team’s theoretical framework of reference. In other words, the modalities of employment (thus also the didactical functionalities) depend on the chosen theoretical framework of reference.

In the following we give a little example.

**Example of didactical functionalities**

Suppose to consider the educational goal of introducing pupils to proofs in algebra. Suppose that a software designer wants to develop a symbolic manipulator in order to achieve such an educational goal.

If the chosen framework of reference is that of the didactic situations theory, then the definition of the didactical functionalities will put peculiar attention on the specific feedback provided by the manipulator consequently to the users’ actions. The designer can for instance suppose that a user tries to transform an expression applying an axiom, then the ICT tool could give information on the correctness of the axiom used; or in other cases the user could freely transform an expression, and the software could give information on what are the axioms underlying the produced transformation. The assumption is that the feedback shell contribute to the learning of the axiom itself, in fact it can be interpreted in terms of retroactions of the milieu consequent to the user’s actions within a specific situation.

On the other hand, if the chosen framework of reference is that of the theory of instruments of semiotic mediation, a particular attention will be put on the nature of the used signs. For instance, like in the case of L’Algebrista (Cerulli & Mariotti 2002), an axiom could be represented in the form of a button including a formula and a name. The “button” would belong to the semiotic system of the interaction user-computer, the formula would belong to the mathematical symbolic semiotic system, and the name would belong to a verbal semiotic system. The interplay between the three different semiotic system can be used by the teacher in order to orchestrate mathematical discussions aiming at building meanings related to the considered axioms, derived from the ICT tool, but expressed and formalized in a mathematical form.

In both cases the designer of the ICT tool, by designing the possible modalities of employment and the characteristics of the tool, provides it with a set of didactical functionalities on the basis of the chosen theoretical framework of reference.

**A model to classify the modality of employment of ICT tools in TELMA researches**

In the previous section the focus was on ICT in general, and we presented a summary of theories that are used by TELMA teams either at the level of development of ICT, and at the levels of choice and use of specific ICT tools. In this section we focus only on the employment of ICT tools in situations of teaching/learning activities. The experimental phase, in fact, turns out to be crucial in order to test the effectiveness of the developed, or individuated, didactical functionalities.

Of course in order to set up an experimentation, and to evaluate its results, it is important to refer to a theoretical framework; for instance several teams (DIDIREM, Did@TIC, NKUA, etc.) refer to the
theory of didactic situations for designing, setting up, and analyzing experimentations, as we will see more in details later on.

TELMA teams employed ICT tools according to quite different modalities of employment. For this reason we developed a model to help us to classify such modalities. The model is named educational experiment cycle, and attempts to describe the basic phases of a teaching/learning activity. Our educational experiment cycle model describe a teaching/learning activity individuating three phases: the planning of the teaching/learning activity; the put in practice of the teaching/learning activity; the diagnostic phase.

![Diagram of Educational Experiment Cycle](image)

**Figura 1 Educational experiment cycle**

Given an educational goal, the planning phase consists of the design and setting up of an activity (or sequence of activities) aiming at reaching the educational goal. The put in practice phase consists of the actual implementation of the planned activity (or sequence). The diagnostic phase consists of some evaluation of the actors involved in the cycle, could they be learners or teachers, with respect to the assumed educational goal.

ICT tools are employed by TELMA teams for educational purposes at any stage of an educational experiment cycle, exploiting their educational functionalities as means for reaching a given educational goal. Moreover, The way how such tools are used in the different phases depend on the specific chosen theoretical framework. For instance, the LINGOT project of DIDIREM employs ICT tools in the diagnostic phase basing its work on the theory of didactic situations, while the ARI-LAB project of ITD-CNR employs ICT tools in the planning phase and in the put in practice basing its work on the Activity Theory.

In the Lingot project, in fact, the DIDIREM team research aims at developing diagnostic and remedial tools in elementary algebra, testing them with students and also studying how teachers appropriate the use of such tools. The hypothesis is that there exists some coherence in student’s behavior, thus understanding this coherence and how it can evolve is a necessity for developing effective diagnostic and didactic strategies based on this diagnostic. Then, the theory of didactic situations is used for supporting the conception of tasks linked to the diagnostic: the definition of the modalities of employment of the used ICT tool, is based on the idea that the teacher submits to pupils a diagnostic activity based on the tool, and the feedback received by the teacher is used as a basis for planning (according to the theory of didactic situations) the tasks to be submitted to pupils in the put in practice activity. In other words the ICT tools is employed in the diagnostic phase of the educational experiment cycle, and the provided feedback contributes to the setting up (planning phase) of the situations to be submitted to pupils, in order to achieve the given educational goal in the put in practice phase. The peculiarity of this perspective is that the ICT tools are employed by the teacher as sources of information rather then as mediators directly fostering pupils learning: the
didactic (or adidactic) situations, planned for fostering learning may even not include an ICT tool at all, even if they have been planned with the aid of a diagnostic ICT tool.

A different example is that of the ARI-LAB project of the ITD-CNR team, which aims developing a software for the arithmetic problem solving, and test it in teaching and learning processes. In this case the ICT tool is used by the team in the planning and put in practice phase. The used ICT tool (ARI-LAB) consists of a set of microworlds and two modes of interaction, the teacher mode, and the pupil mode. In the pupils mode it is possible to interact with the software solving tasks within one, or more, of the available microworlds. Not all the developed microworlds are always available to the pupils, in fact in the teacher mode it is possible to set up tasks to be submitted to pupils, and for each task it is possible to chose which specific microworlds can be accessible to the pupil in order to solve the task. In other words the modalities of employment of the ITC tool involve both, the planning and the put in practice phase.

In particular, in the planning phase the ARI-LAB can be used by the teacher to setup an activity (aimed at developing certain arithmetical competencies) in terms of defining the characteristics of the microworlds available to the user. The theoretical framework of reference is that of the Activity theory, which is used for supporting the definition of a context of use of the tool in terms of considered activity, configuration of the ICT tool, participants to the activity, social rules, etc. In the put in practice phase, learning is assumed to be an outcome of the planned activity which involves among other elements, the pupils and the ICT tool. As a consequence, configuring the tool, is a way, for the teacher, to define specific didactical functionalities as means for achieving her specific educational goals.

### The theoretical frameworks of TELMA teams

#### DIDIREM team

**Overview**

DIDIREM team research in the area of technology and mathematics education relies on different theoretical frames not necessarily specific to technology, and contributed to the development of a more specific one, known today as the 'instrumental approach'. This approach developed jointly by M. Artigue, J.B. Lagrange, D. Guin and L. Trouche relies both on ergonomy and anthropology. It provides a frame for analysing the processes of instrumental genesis both in their personal and institutional dimensions, and the effect on instrumentation issues on the educational integration of computer technologies.

The start point of Didirem team research is a corpus of publication from 1994 to 1998 about research and innovation in the world-wide field of integration of ICT (Lagrange, Artigue, Laborde, Trouche, 2003). The aim of this paper was to study the issues of the integration of technology into mathematics teaching to better understand the difficulties of integration, and to review the potential and limitations of existing researches.

The complexity of ICT integration is evident, in particular in France where difficulties are persistent in spite of the continuous governmental support given to integration for more than 20 years now. Moreover, during the last two years an extensive survey of technology in mathematics teaching (Lagrange et al., 2000, 2001) took into account more than 600 publications and reports, published between 1995 and 1998, of which 175 dealt with CAS. The results clearly show that the complexity of instrumental genesis has been widely under-estimated in research and innovation on TICE, until quite recently.

**Aims**

The main aims of this research are:

- understanding the dialectic relationship between conceptual and technical work in such technological environments,
• understanding better the ways technological objects can become mathematical instruments for the students and the teacher;
• approaching these issues without underestimating the institutional dimension of learning processes in school.

ICT tool
DIDIREM uses mainly professional ICT tool such as CAS and spreadsheets.

Theoretical framework of reference
The Theoretical Framework of reference used by the team are the following:
- the instrumental approach
- theory of didactic situations

DIDIREM team research uses the 'instrumental approach'. This approach relies both on ergonomy and anthropology. The ergonomic theory is employed by the team refers to analyse the interactions between learner and CAS. The anthropologic theory is used to situate such interactions within an institution; in fact a CAS is in general meant to be a professional tool, not designed for being used in class. An anthropologic theory is thus needed in order to identify the didactical functionalities of the CAS, which may be not indicated by the producers of the tool. The theoretical concepts of "practical techniques and epistemic techniques" allow the study of the relationship between conceptual and technical work in such technological environments.

This research about CAS of DIDIREM focus on the whole educational experiment cycle which is developed not only on the basis of the instrumental approach but also on the basis of the theory of didactic situations.

The theory of didactic situations is used to analyse the characteristics of learning environments involving technology (CAS), and infer from these characteristics what mathematics can be learnt and how in these environments. In this case the theory is used in the planning phase of educational experiment cycle.

The concepts of “fundamental situation” and didactical variable, are instead needed in order to define a situation allowing the development of a good diagnostic. In this case the theory is used in the diagnostic phase of educational experiment cycle.

Example
In the Lingot project, the team research aims at developing diagnostic and remedial tools in elementary algebra, test them with students and also study how teachers appropriate such tools. The hypothesis is that there exists some coherence in student’s behavior and understanding this coherence and how it can evolve is a necessity for developing effective diagnostic and didactic strategies based on this diagnostic. Then, the theory of didactic situations is used for supporting the conception of tasks linked to the diagnostic. Diagnostic had been computerized during a previous project (Pepite project). The main aims of the diagnostic computerization are to make the transition from a fixed to a flexible and dynamic tool of diagnostic, to develop paradigmatic classes of situations for remediation depending on parameters corresponding to didactic variables and to develop a module organizing the connection between the diagnostic and remediation tools.

Educational technology Lab, NKUA Greece

Overview
The general research area of the team is to study learning, teaching, contextual and organizational processes during the infusion of innovative educational environments in everyday practice. The main interest concerns the kind of cognitive, social and systemic turbulence created as a result of questioning and change in everyday practice. Team is engaged in investigating ways in which this turbulence might be channeled so that students generate mathematical meanings, teachers develop
strategies for creating environments where this is likely to happen, and organizations accept, recognize and value such change.

**Aims**

The main aims of this research are:

- understanding how the generation of mathematical meanings for students può avvenire and how they may be influenced by the classroom norms,
- understanding the role of the teacher in the classroom and how his/her work changes if an ICT system is introduced in the teaching and learning practice

**ICT tool**

ICT systems used by NKUA team are E-slate microworlds. E-slate is a construction kit for educational software, addressing both technical and non-technical users. It is made up of components, which are pieces of software designed to be as generic as possible, a custom made desktop environment providing services for construction, layout and use, and two distinct ways of connecting components, that is, either using prefabricated connections or defining your own through a programming language.

Design and development of software and activity plans for meaning generation involves a variety of areas, such as language, history, programming, mathematics, science and geography. NKUA team is interested in particular in mathematics areas.

**Theoretical framework of reference**

The main theoretical aspects of reference used by the team are the following:

- ‘situated abstractions’ (Noss and Hoyles, 1996) to study generation of mathematical meanings for students (vedi Microworld theories)
- classroom norms (Cobb & Yackel, 1996) (vedi Appendice)

The NKUA team study the ways students use representations to express mathematical meaning in E-slate microworlds.

In order to understand student meaning generation the team has found the construct of ‘situated abstractions’ (Noss and Hoyles, 1996) a useful tool: generation of mathematical meanings for students is studied by the team when students interact with a technological environment.

The uses and roles of such technological environment have been influenced by the paradigm shift from constructivism to socioconstructivism.

“The initial notion was that learning will be realized simply by giving children opportunities to interact with rich computational environments, to work with open-ended exploratory soft-ware (Papert, 1980). Research evidence (Kafai & Resnick, 1996) indicated that the computer has the potential to overturn many of the assumptions about what children can and cannot do, the hierarchies of understanding that have been drawn up so far, and the ‘readiness’ of pupils to understand this or that mathematical concept (diSessa et al., 1995). However, such research evidence also indicated that, simply by interacting in an environment, children are unlikely to come to appreciate the mathematics lying behind its pedagogical intent (Noss & Hoyles, 1996).”

(Kynigos C. & Argyris M., 2004)

The socioconstructivism paradigm is strictly connected with the teacher’s role.

“The idea is that students construct their knowledge and understanding of the world not just through direct personal experience and discovery (constructivist paradigm), but also through the intellectual sharing and support of those around them (socioconstructivist paradigm). In these terms, the role of teacher is a particular, skilled form of such support (Mercer, 1993). This gives impetus to classroom research with particular emphasis on the interplay between mathematical learning and social interaction (Cobb et al., 1997). Such studies involves perceiving the teacher in terms of her/his role in organizing the class, setting up tasks for the students to be engaged in and supporting their learning process. The notion of the teacher influencing the construction of classroom norms (taken as shared meanings and understandings of actions and roles) is crucial to this perception of the teacher (Cobb & Yackel, 1996).
This framework for perceiving the teacher thus involves a reorientation of the ways in which the nature of the teaching profession is addressed. It points to the conception of teaching as a thoughtful profession, rather than a technical mediation of a prescribed curriculum and to teachers as professionals who have an active role play in planning curriculum, designing and supporting educational activities (Jaworski, 1994, 1998).

Such a stance is based on two assumptions (Fang, 1996). First, teachers are professionals who make reasonable judgments and decisions within a complex and uncertain community, school and classroom environment. Second, teachers’ thoughts, judgments and decisions influence their classroom behavior. In line with this notion, teacher research has made significant strides, studying the complex relationships between beliefs and classroom practice (Grant et al., 1998; Aguirre & Speer, 2000). Distinctions like the one between beliefs and beliefs-in-practice, termed ‘situated beliefs’ by Hoyles (Hoyles, 1992), have emerged providing useful insights. The perception of teachers making sense of their environment as they act upon it, constructing and reorganizing a personal pedagogy through interrelation with classroom and school culture, has come into focus (Lerman, 2000).

Besides these perceptions of ‘the teacher in the classroom’, however, little attention has been given to the systemic level; that is, the ways in which aspects of the educational system and the educational paradigm as a whole may influence the types of things teachers take for granted. In central European countries, for instance, and particularly in Greece, there is a strong element of revelation (of knowledge, or of the truth) engrained in the curriculum (Kontogiannopoulou-Polidorides, 1996), part of the cultural heritage of the past century. It may be the case that this would make it much more difficult for a teacher working in this system (rather than within an Anglo-Saxon one) to understand, adopt and value innovation where students are encouraged to generate their own meanings through personal expression, experiment and constructions” (Kynigos C. & Argyris M., 2004).

The research of the team with respect to teacher practice focus on the teachers’ role and communicational modes with their students in an attempt to identify the kinds of beliefs-in-practice (Hoyles, 1992) emerging from participating in technology-based innovational work. The teacher is perceived as a professional who constructs and reorganizes a personal pedagogy through interrelation with classroom culture and the wider culture, acting as a mediator.

This team studied the ways students express mathematical meaning in an integrated way in technological environments. The attention focus on how the use and the affordances of these interacts with students’ mathematizations in problem situations. This study concerns student meaning generation for a range of concepts, such as orientation and spatial awareness, angle, properties of geometrical figures, ratio and proportion – similarity, sin – periodicity, tangent – non-linear functional relationships.

Example

As an example of situated abstraction, we report a study, elaborated by the team, on meanings of scale generated by pairs of 14 year-old students engaged in joint map-construction. The students worked in a cartography microworld of E-slate which combines symbolic expression to construct figures with dynamic manipulation of the generated graphical output. This study aimed to understand weather and how students engaged in proportional reasoning during their attempts to construct the contours of the several buildings of their school site and their respective positions, orientations and relative sizes, so that the figural relationships of these representations would be preserved when dynamically changing the scale.

The study was centred on the evolution of students mathematical meaning about proportional relation. Initially students adopted a “componential” approach, involving meaningful constituent parts of a map. For example, they measured each segment of a building’s perimeter using a specific unit (amongst the units chosen were a foot, a step, a belt and a meter). In this work they are not able to coordinate building sizes. When they tried to enlarge the building using a specific tool of the microworld, they ended up distorting the building. Apart from relations between buildings and side lengths, spatial relationships also involved distances between buildings. This became problematic when the students realized that dynamic scale changes of maps containing two buildings distorted the distance between the buildings and thus their topological positioning on the map.
Only subsequently, students adopted a holistic view of the map, realizing that topological relations involved both the objects themselves and their relative positions. In this evolution of students mathematical meaning about proportional relation, it is possible to see the framework of situated abstractions as proposed by Noss and Hoyles (1996). Abstraction is a process which develops in activity, and from this perspective, the process of abstraction can be seen as a way of layering meanings on each other. In this study, the process of abstraction which allows to give a meaning about proportional relation for the whole map layers to the meaning about proportional relation for the single entity of the map.

**ITD-CNR of Genova**

**Overview**

The work carried out by ITD-CNR team is concerned mainly with the design, the implementation and the experimental evaluation of innovative ICT based systems for mathematics learning at school level. The research work is characterised by an integrated approach where tools are designed and studied as embedded in educational contexts which are themselves objects of the research.

**Aims**

Research developed by ITD-CNR team is aimed at studying how new technologies, if inserted in suited contexts, can contribute to the construction of innovative environments that can enhance learning processes and can also change traditional approach to school teaching.

**ICT Tools**

ITD-CNR team has been involved in a long-term project, the ARI-LAB project, that is oriented to the design, development, in-field experimentaition, and critical evaluation of a multi-environments system (ARI-LAB) for maths education at compulsory school level (pupils’ age from 7 to 12). The project has evolved during times and has brought to the development of different versions of the ARI-LAB system. The more recent developments of ARI-LAB project are the two systems ARI-LAB-2 and ARI@ITALIES. ARI-LAB-2 is a system for arithmetic problem solving designed to be used in a class, or in a school computer laboratory, supporting computers connected through a local network. ARI-LAB-2 allows the user to interact with a structured and interconnected set of environments which includes microworlds, data-bases of solved problems, an environment for building problems solutions, and a communication environment. The ARI@ITALIES system features a kit of tools (partly based on ARI-LAB-2 environments) to be used by teachers for building components of online courses on curricular math topics at primary and lower secondary school level. It allows the teacher to build interactive and constructive activities and the student to develop them through a normal web browser.

**Theoretical frameworks**

The main theoretical aspects of reference used by the team are the following:

- *Activity theory* (Engeström, 1987; Cole and Engeström, 1991)
- *Microworld related theories*
- *situated abstraction* (Noss and Hoyles, 1996)

The ARI-LAB project is based, on the one hand, on research on interactive visual microworls, on communication systems and on knowledge representation. On the other hand, it takes into account the research in mathematics education with particular reference to the studies in the domains of arithmetic problem solving and introduction to algebra, to the role of visual interactive representations, and to studies related with learning in social contexts.

The initial design of ARI-LAB started from an epistemological analysis of the nature of knowledge involved in arithmetic problem solving with particular attention to the study of the pedagogical difficulties and obstacles that students often encounter in solving arithmetic problems. Reference was made researches that stressed the dangers of early formalization and undue stress on written
computation, focusing on the importance of developing concrete meanings for symbols and formal procedures (see, for example, Lesh, 1985; Nesher, 1986, Schliemann, 1995).

In the design of ARI-LAB microworlds reference has been made to such studies as well as to the notion of field of experience developed by Boero in (Boero et Al., 1995). This work pointed out the importance to develop activities related to unitary and homogeneous sectors of human culture that are meaningful both for the students and for the teacher (for example, the fields of experience of “purchase and sales”, and that of “calendar” have been used as reference to model two of the microworlds of ARI-LAB). In the log run, arithmetic itself may become a field of experience. This understanding confirm the opportunity, when designing a ICT-based educational tool based on microworlds, to provide a wide-ranging set of environments that can support the evolution of mathematical knowledge embedding both microworlds related to concrete, real world activities and microworlds where mathematics activity can be progressively expressed in more formal way.

At the basis of the concept of microworld there are the objects of the interface that it makes available to the user. The analysis developed by Hoyles (1993) on situated abstraction has supported our understanding that carefully designed computational objects can play a crucial role in approaching mathematical knowledge. As a matter of fact, they allow the exploration and the manipulation of concrete (visual, motor perceptive...) representations through which students can grasp and develop meanings for related abstract maths concepts. From a theoretical point of view, such perspective is related to the analysis of the mediating role of tools in teaching and learning processes that ITD-CNR team derived from Vigotsky. Vygotskij (1962, 1978) and many others have stressed that languages (regarded as symbolic systems) may not only represent but also direct human thought. In this regard, representations that are not completely arbitrary but preserve some analogical link with the related objects (such as geometrical figures, histograms, line of numbers, coins) have proved very important. The work with visual representations differs from the manipulation of sets of objects just because of their status of symbolic systems with a recognisable set of rules. Languages of this kind act as mediators between the problem situation and its meanings and the mathematical ideas, relationships and processes involved.

This process of mediation is fundamental in shaping the internal cognitive structures of the individual if included in a context of social interaction that can provide a validation to individual strategies and the dialectical construction of meanings for the whole activity enacted. By ‘context of social interaction’ ITD-CNR team mean all kinds of cultural exchange between the student and the environment (teachers, other students, …) even through the mediation of various devices (books, computers, board, …).

In our work, the acknowledgment of the crucial role played by contextual aspects in ICT mediated teaching and learning processes has been strengthen by the iterative process through which ITD-CNR team have developed the different versions of the ARI-LAB system. In such process a crucial role was played by the in-depth class experiences ITD-CNR team made in several different situations. This experimental work has helped us to focalise on the social nature of cognition and meaning (Resnick, 1987) and to adopt Salomon’s olistic view of learning environment (Salomon, 1996). According to this perspective, the term learning environment is not used to describe merely a software tool, as often red in the literature, but something that encompasses the whole context within which technology is being put into use.

In building a meaningful educational environment, a software tool can have an important role as an artifact mediating teaching and learning processes, but it is only one of the components of the whole environment. Not less important are the pedagogic activities in which the use of the tool is integrated, the way in which these activities evolves, the social interactions that take place, and the way in which the work is organised and embedded in the general structure of the school and of the educational institution. This is particularly true when technology is being studied in relation with long-term teaching and learning processes of the kind needed for the development of complex articulated abilities such as those involved in arithmetic problem solving. For the development of such abilities, the student-software unit of analysis is not sufficing as it is necessary to consider the
whole set of interactions established in a class over the course of time. As the matter of fact, the mediation offered by a given software to cognition is not sufficient to explain the learning aspects related with motivation, with goals formation and with the attribution of a meaning to the whole activity which goes beyond the meaning of the single actions involved in the performance of a task. The analysis of these aspects requires looking at learning not only as an individual construction developed during the interaction with the computer but also as a social construction developed within the whole learning environment.

It thus emerges the crucial role of context in the construction of meaningful learning environments and the necessity to study the influence of contextual aspects on concepts understanding and meaning construction (see Bottino e Chiappini, 2002).

The attention to the learning environment as a whole has brought us to progressively refer to theories that highlight the importance of studying the relations among individuals, mediating tools, and the social groups. ITD-CNR team have made reference, in particular, to Activity Theory that gives us a framework, namely terms and notions associated to those terms, that ITD-CNR team considered useful for analysing the learning environment where the ICT tools ITD-CNR team have developed are integrated. Referring to Activity Theory, ITD-CNR team interpret a learning environment as constituted by the enactment of a teaching and learning activity oriented to an educational object, involving students, teachers, and tools. A learning environment is not something that is assigned or planned in advance, but it is negotiated and built by participants in the enactment of a teaching and learning activity and evolves during its development.

Activity Theory has given us a reference for making explicit and for analysing the main components that shape technology mediated learning environments, and has suggested a way to examine how such components interrelate. More specifically, ITD-CNR team use the Cole and Engeström model of the complex relationships between elements in an activity (Cole and Engeström, 1991). This model allows us to develop a methodology for performing the analysis of the learning environments where ARI-LAB is integrated (see activity theory in annexes).

ITD-CNR team has defined a methodology for analysing learning environments mediated by technology through the identification of three main directions of analysis on the basis of the Cole and Engeström’s model of activity. The three directions identified are the following:

- How the educational technology used can mediate new ways for the learner of accessing, representing, and interact with the concepts, procedures, and rules that are involved in the acquisition of a given mathematics knowledge which constitutes a learning object for a teaching and learning activity.
- How the educational technology used can contribute to the design and the enactment of didactical practices aimed at an evolution in the use of the rules related to the knowledge to be learnt and to the construction of appropriate ways and meanings for using them.
- How the educational technology used can contribute to mediate the assumption of new and old roles by participants in the didactical practice.

University of Pisa

Overview

This team focuses on the use of technology in mathematics education. Different mathematics domains (Algebra and Geometry), and till now different microworlds have been considered, but with a common theoretical assumption. According to a vigtokian perspective, the research studies are centered on the notion of “semiotic mediation”: specific elements available in the microworld can function as instruments of semiotic mediation in relation to the construction of the mathematical meanings which constitute the educational goals.

This research can be classified as research for innovation: theoretical and experimental studies maintain a dialectic relationship. A first set of research hypotheses frames the design of the teaching experiments, results, coming from those experiments, contribute to the evolution of previous hypotheses and consequently to the design of new experiments.
Aims
The main aims of this research are:
- understanding how the meanings rooted in activities involving artifacts can evolve toward mathematical meanings under the guidance of the teacher by means of peculiar semiotic practices
- individuation and definition of the above mentioned semiotic practices
- understanding what characteristics of an artifact can be better exploited as instruments of semiotic mediation

ICT tools
The Pisa teams conducted its researches basically using educational ICT. The team has been involved both in studying existent ICT, like in the case of Cabri, and in designing realizing and experimenting new ICT, such as in the case of L’Algebrista.

Theoretical framework of reference
The main theoretical aspects of reference used by the team are the following:
- Semiotic mediation (Vygotskij, 1978)
- Activity theory
- Rabardel’s theory
- Microworlds and related theories
- Theory of instruments of semiotic mediation (described below)

This team developed and employed the theory of instruments of semiotic mediation with respect to two particular "fields of experience" (Boero et al., 1995), strictly related to the use of a didactic software: the "geometric constructions in the Cabri environment" (Mariotti, 1996; 1998; 2001); and the field of “proving equivalencies of algebraic expressions in the L’Algebrista environment”.

The study is based on the theoretic framework of Vygotskij, with particular attention to the social construction of knowledge. The main findings on which the proposed research is founded concern the process of semiotic mediation related to some particular cultural artefacts, i.e. software environments or ICT.

According to the Vygotskian framework, different tools, provided by the microworlds, can be thought as external signs referring to a specific mathematical meaning, as such, they may become tools of semiotic mediation (Vygotskij, 1978). Meanings are rooted in the phenomenological experience (actions of the user and feedback of the environment, of which the artifact is a component) but their evolution is achieved by means of social activities.

Thus, according to this perspective, the role of the teacher is crucial; besides the organization of the teaching environment (for instance selecting the task to be proposed), the teacher can choose specific tools of semiotic mediation to be used in social activities in order to guide the evolution of personal meaning towards the mathematical meaning which constitute the motive of the teaching/learning activity.

Among others social activities of the class, a special role is played by 'Mathematical Discussions' (Bartolini Bussi, 1996). It is expected that the analysis of verbal interaction, realised in mathematical discussions, reveals specific communication strategies which can be used by the teacher in order to foster the construction of mathematical meaning in relation to specific semiotic tools.

Examples
A synthesis of how Cabri-Géomètre has been used as an instrument of semiotic mediation in the context of geometry is given by Cerulli in his PhD thesis:

“Mariotti set up and carried out a long term teaching experiment concerning geometrical construction in the “ruler & compass” and the Cabri environments, aiming at introducing pupils to axiomatic Euclidean geometry, and more generally to theoretical thinking.
Given such a didactic problem, the principal motive of classroom activities is centred on the evolution of the idea of proof, and is realised "by means of the evolution of the idea of geometrical construction, within the field of experience of geometrical constructions in the Cabri environment (Mariotti et al, 1997; Mariotti, 2001)". The author introduces the term "field of experience" referring to:

"the system of three evolutive components (external context; student internal context; teacher internal context), referred to a sector of human culture which the teacher and students can recognise and consider as unitary and homogeneous" (Boero et al., 1995)

Within the approach presented by Mariotti, there are two main interlaced fields of experience, that of geometrical constructions in the Cabri environment, and that of geometrical constructions in the paper and pencil environment; according to the author, the field of experience can evolve over time thanks to social practices of the classroom. Among such practices, she focuses on the verbal interaction realised by means of "mathematical discussion" as defined by Bartolini Bussi:

"a polyphony of articulated voices on a mathematical object, that is one of the objects - motives of the teaching - learning activity” (Bartolini Bussi, 1996).

In the experiment presented by Mariotti, the polyphony of voices concerns a dialectic between the voice of practice, and the voice of theory, that is, a practical conception of graphical construction versus a theoretical conception of geometrical construction (Mariotti, 2002)

In the case of the production of a drawing on a sheet of paper, its validation is demanded by the empirical verification of a practical objective, whilst geometrical constructions have a theoretical meaning overcoming such a practical objective. Geometrical constructions are based on theorems that guarantee theoretical control on the correctness of the procedures followed to realize the constructions themselves. Such theoretical control is not spontaneously achieved, but can be fostered by the activities performed by pupils in the Cabri field of experience:

“As experimental evidence shows, theoretical control is not spontaneously achieved, but can result from the activities that pupils perform within the chosen field of experience.” (Mariotti, 2002)

According to Mariotti, the nature of the Cabri environment, may foster a shift from the practical to the theoretical meaning of geometrical constructions, nevertheless the environment itself is not enough, and the intervention of the teacher becomes determinant. However, some elements of the Cabri environment are presented as key elements for the development of a dialectic between the practical and the theoretical level, in fact they can be interpreted as external signs, standing for elements of a geometrical theory:

“- the primitive commands and macros, realising the geometrical relationship characterising geometrical figures, are the external signs of the basic elements which constitute the theory;
- the dragging function which starts as a perceptual control tool to check the correctness of the construction, then becomes the external sign of the theoretical control.” (Mariotti, 2002)

When geometrical activities are concerned, these elements of the software, are viewed as the external signs on which “the evolution of pupils' internal context is based”; such evolution concerns the development of both, the aimed geometrical theory and at the meaning of theory itself.

In the teaching experiment, presented by Mariotti, the meaning of geometrical construction emerges from activities of construction, within Cabri, and from related mathematical discussions; through the practice of mathematical discussion, the way pupils make sense of the construction activities within Cabri, is elaborated and developed under the guidance of the teacher.” (Cerulli 2004)

The framework pays particular attention to the chosen educational aims, which in this case (introduction of pupils to geometrical constructions) is strictly related to the introduction of pupils to geometry as a theory. In the following we describe how Cabri has been used as an instrument of semiotic mediation in order to introduce pupils to the geometry as a theory:

“According to Mariotti, two, strictly interwoven, areas of difficulty can be individuated when the idea of proof is concerned. The first is that the idea of validation must be introduced, the second is that the rules for validation must be stated, and their acceptance influence the acceptance of the idea of validation. Within the approach to geometry described by the author, the basic aim was that of introducing pupils to theoretical thinking and in particular to Geometry theory, as a consequence it was decided to build and exploit a dialectic relationship between geometrical theorems and Cabri constructions:

“Starting in the Cabri environment pupils should have been guided to enter into the geometrical system, the key of access was the link between the logic of Cabri, expressed by its commands, and the Geometry Theory expressed by its axioms and theorems.”(Mariotti, 2002)
However, the software offers many commands, a richness which might foster the ambiguity about intuitive facts and theorems, and constitute an obstacle for pupils to grasp the meaning of proof. As a consequence, in the approach presented by Mariotti, pupils are presented at first with a limited set of commands corresponding to a limited set of Euclidean axioms. Along with the development of the activities, pupils could build their own menus adding commands, corresponding either to axioms of the Euclidean theory, or to new constructions which corresponded to new theorems:

“Taking advantage of the flexibility of the software environment the microworld was adapted to follow the evolution of the theory: at the beginning, an empty menu was presented and the choice of commands discussed, according to specific statements selected as axioms. Then, in the sequence of the activities, the other elements of the microworld were added, according to new constructions and in parallel with corresponding theorems.” (Mariotti, 2002)

As a consequence, pupils may be guided to slowly build up a geometrical system, in so doing they cope with a complexity they can manage, but at the same time, they participate in the construction of menu and it corresponding axiomatisation. The author reports evidence of the fact that in such kind of experimentation, the construction problem can achieve, for pupils, a theoretical meaning, while the commands of Cabri can be transformed into signs of the theoretical control corresponding to axioms and theorems.” (Cerulli 2004)

The same theoretical framework has then be used for the L’Algebrista project in two different phases, in the realization of the software, and in its experimentation, contributing to a further elaboration of the framework itself. The framework in fact has been used in order to define the characteristics and the educational functionalities of the software L’Algebrista with respect to the educational aim of introducing pupils to symbolic manipulation and to algebra as a theory.

MeTAH and Leibniz-IMAG, Did@TIC team

Overview
Did@TIC team works for long time on design computer system encompass APLUSIX project concerning, among other things, modelling in ICAI (Intelligent Computer Aided Instruction). Did@TIC team developed, on the bases of the resolution theory elaborated by Nicaud, the APLUSIX system, a computer program that helps students to learn algebra.

Aims
The main aims of the Did@TIC research are the following:
- design and development of Aplusix learning environment for algebra;
- analysis of the use of Aplusix System
- analysis of student’s modelling in algebra (second degree schools).

ICT Tool
Did@TIC team si è occupato prevalentemente dello sviluppo di un learning environment per l’algebra: Aplusix

Theoretical Frameworks of reference
The Theoretical Framework of reference used by Did@TIC team is based on the field of Mathematic Education (Didactique) and on the field of Artificial Intelligence.

In particular, the following theories are brought into account:
- Theory of didactical situations (Brousseau, 1997),
- cKé (Balacheff, 1995) that isa theory supporting didactical aspects of the research (see annexes)
- Artificial Intelligence theories (Rewriting rules theory and Resolution theory, see annexes) that give mechanisms for solving problems.

The research brought forward by Did@TIC team involves both a phase of design and development of the system Aplusix, and an experimental phase concerning the system and its didactical functionalities concerning algebra. The two phases are interlaced because both, design and use,
contributed to the evolution of the system. In fact, the design of the system evolved on the basis of the feedback obtained from the *Educational experiment Cycle*. 

In order to describe the relationship between the two phases (design and use), the Did@TIC team refers to the theory of didactic situations. This theory considers that a new knowledge is the result of a construction made by the learner in interaction with a *milieu* organized by the teacher. Aplusix is considered as a *milieu* for learning algebra. The theory of didactical situations, gives tools to characterize learning environments in terms of possibilities of actions and feedback offered to the student: the student acts on the *milieu* and the modifications of the *milieu* provide feedback that can provoke the desired learning. Feedbacks provided by Aplusix are, for example, validations of the calculations and some particular indicators. 

The observation and the analysis of the interactions between subject and milieu, during the experimentation, allow the researchers to re-elaborate the design of the system identifying some missing *didactical functionalities*. In this sense the theory of didactic situations is used as a means for re-elaborating the design and for producing new versions of the system. 

In this approach, the learning is a modification of the relationship to the knowledge produced by the student in interaction with the *Aplusix*. 

For modelling the knowledge state of the learner in Aplusix, Did@tic use the *cK¢ model*. This model proposes to characterize a conception C by a quadruplet (P, R, L, $\sum$) in which P is a set of problems on which C applies; R is a set of operators, L is a representation system, and $\sum$ is a control structure. 

The model allows to implement in an ICT system the components characterizing a set of conceptions so that it is then possible to individuate which of such conceptions are employed by the user during his/her interaction with the system during the resolution of a problem. 

For what concerns Artificial Intelligence, Nicaud develops a resolution theory (Appendice), taking into account the following theory: 

“Several general problem solving methods have been defined in Artificial Intelligence, in particular for problem domains where objects and operators for transforming the objects are defined. Algebra enters this framework by considering that the objects are the algebraic expressions and that the operators are the transformation rules. 

[…] Among these methods the more pertinent for human algebra is that called heuristic search. This method solves a problem by developing a search graph using a forward process that applies operators to states (the nodes of the graph). It does not suppose the termination of some class of operators on solved states and, as a consequence, it uses backtrack as a fundamental concept. Heuristic search makes use of domain specific knowledge for choosing the operators. As it does not suppose a precise goal (it does not suppose that a solved state is given), it is more pertinent for human algebra” 

(Nicaud. 1994) 

This theory, together with the Rewriting Rules theory (see annexes), can contribute to elaborate new resolution theory for the polynomial factoring domain. This domain is the base of Aplusix concepito e implementato for writing algebraic expressions and reasoning. 

**Institute of Education, University of London** 

**Overview**

The theoretical framework on which Institute of Education team works is based on the Social Semiotics (Halliday, 1978). This team investigates about the uses of language in teaching and learning practices of mathematics in order to develop knowledge about the language as tool to favorire the teaching and learning processes in mathematical domain. 

**Aims**

The main aims of the research team are the following:

- Studying the nature of mathematics and mathematical activity as it is constructed in a text
• Studying who are the author and reader and what relationships do they have to each other and to the subject matter
• Studying the role which the text plays within a context of a particular situation (to tell a story, construct a description, give a set of instructions for a procedure, make an argument)

ICT Tool
The team actually doesn’t use ICT tools in its research. However, in the context of the TELMA research, the team proposes to use ICT tools which dispose of communication functions (such as chat …). These functions should allow students to produce texts which can be analysed by means of the theoretical tools of the research team.

The introduction of ICT tools modifies the aims of the research team. As a way of analysing texts and the reading of texts the social semiotic framework gives a way of understanding the ways in which students may make sense of what is presented to them by technological tools. This theoretical framework is also a means of making sense of classroom interactions more generally and the place that technology may play within a classroom context. For example, the team proposes to use the social semiotic framework to look at the different kinds of communications that go on when pairs of students work on problems using dynamic geometry when they are alone and when a teacher is present.

Theoretical framework of reference
The Theoretical Framework of reference used by the team is an elaboration of the Social Semiotics theory of Halliday (1978).

The Institute of Education team argues that Halliday’s theory of language as social semiotic (Halliday, 1978; Halliday and Hasan, 1989) and the associated tools of systemic functional linguistics (Halliday, 1985) provide some powerful ways of investigating the practices of teaching and learning mathematics. This theory allows the team to develop knowledge about uses of language within mathematical practices that may be helpful for teaching and learning.

About language as social semiotic Morgan says that

“At its most basic level, a social semiotic perspective involves recognising that language consists of “the exchange of meanings in interpersonal contexts of one kind or another” (Halliday, 1978, p. 2) and that this exchange of meanings is functional. Individuals do not speak or write simply to externalise their personal understandings but to achieve effects in their social world. Studying language and its use must thus take into account both the immediate situation in which meanings are being exchanged (the context of situation) and the broader culture within which the participants are embedded (the context of culture).

The context of situation encompasses the goals of the current activity, the other participants, the tools available and other aspects of the immediate environment. Each situation cannot be considered in isolation but as an example of a situation type or semiotic structure formed out of the sociosemiotic variables: field, tenor and mode. The field of discourse may be thought of not simply as the subject matter but as the institutional setting of the activity in which a speaker and other participants are engaged. Tenor encompasses the relationships between the participants, and mode refers to the channel of communication (e.g., writing or speech) and other aspects of the role of language in the situation.

The context of culture includes broader goals, values, history and organising concepts that the participants hold in common. This formulation of context of culture suggests a uniformity of culture both between and within the participants

[…]

The context thus provides the semiotic structure within which exchange of meaning takes place, but to study meanings within a particular situation also requires tools for examining the communicative exchange itself – the language. There are two fundamental characteristics of Halliday’s linguistics: the notions of system and function. Within a given situation, there is meaning potential associated with the type of situation, constituted by a system

1 Candia Morgan uses the terms author and reader, which are normally used in the context of written texts and suggest univocal text. They could be substituted by speaker and listener or, more generally, by correspondents to allow consideration of spoken and dialogic, multi-vocal texts.
of semantic options from which speakers choose. […] It is structured according to the functions that the language is being used for within the situation.

There are three particular functions:

The **ideational** function is the semantic realisation of the field of discourse. That is, the expression of meanings related to the content of the situation, the objects, participant structure, actions and logical relationships between these.

The **interpersonal** function is the realisation of the tenor of discourse. That is, the expression of meanings related to relationships between the participants and to the identity of the speaker.

The **textual** function is the realisation of the mode of discourse. That is, the way in which language itself is playing a role within the situation.

These functions are represented in texts by different parts of the lexico-grammatical system. The relationship between situation type and semantic system allows us, in a very general and non-deterministic way, to predict in both directions. In other words, given a situation, we can predict the types of things that are likely to be said by participants and, conversely, given a text, we can predict the type of situation in which it is likely to have arisen.

[…] The relationship between situation type and semantic system allow the this team to predict, given a situation, the types of things that are likely to be said by participants, and, conversely, given a discourse, to predict the type of situation in which it is likely to have arisen. In this sense, this relation is a tool for examining the language in the communicative exchange.

**Example**

We give an example of “system of meaning” for the role of “reader” which can assume for two different speakers in a same situation.

The analysis developed by the team deal with the investigation of two students’ writing. The writings were produced for examination purposes by secondary school students in England. The text are in the form of reports of mathematical investigative work on a task entitled “Inner Triangles”. Students had to drew trapezia with various dimensions on isometric paper and construct tables to records the dimensional and the number of unit triangle for each trapezium.

The texts of the two students, Steven and Clive, both responding to the same problem, thus construct very different images of the objects of mathematics and the nature of mathematical activity. At the same time they claim different types of authority and construct different ‘ideal’ positions for their readers.

When Steven presents results written in a table to a potential reader, the reader is invited to observe the pattern for themselves (As you can see …). The positive modality of this address to the reader, plays an important interpersonal role, building an image of Steven as authoritative and constructing a reader who is expected to be interested in being informed about what Steven has found.

On the contrary, when Clive presents results written in a table, he writes some claim in a impersonal way (Below the table show the results…; here is another conversion table…) We can say that in Clive’s the relationship between the author and his reader is described as between student and examiner. In displaying his results, there are no suggestions that their mathematical content might be of interest in itself.

The different positions of the two students for the role of the “reader” shows how different systems of meanings can be selected by the participants of the situation type.

Moreover, Steven draws on a discourse of investigation that values exploration of interesting mathematics while Clive draws on an assessment discourse that values answers.
frame their work at the theoretical and practical levels and to justify solutions proposed by their research program.

For example, the notion of learning environment is a topic of crucial interest in the researches of all the partners. We observe that different theoretical frames are employed:

- to define what is a learning environment;
- to analyse its role in the learning of mathematical objects and in the development of cognitive functions associated to the command of such concepts;
- to justify the choices which structure in a particular way a learning environment;
- etc.

Other problematic issues of common interest, which are addressed by TELMA teams using different theoretical frameworks are the following:

- The relationship between teacher and students in teaching and learning processes;
- The role of instrument in teaching and learning processes.

We believe that the comparisons between these problematic issues of common interest as they are framed in different theoretical frameworks, can be a useful tool to facilitate an integrated approach (to ICT tool in Mathematics educations) among TELMA teams.

**Learning environment as a problematic issue of common interest**

In order to consider the use in context of software for mathematical education, an important concept which raises in the works of the TELMA teams, is the notion of learning environment. This concept evolved, in the last decades, on the one hand because of the evolution of technology and its role in educational practices, and on the other hand because of the evolution of the assumed frameworks of reference.

In the past, often the introduction of Information and Communication Technologies (ICT) in educational activities had been linked to a vision of learning as an individual process whereby knowledge emerged from the interaction between the student and the computer. This vision can be inferred from the terminology frequently adopted in the literature, where educational software applications were often referred to as learning environments, thus focusing attention on the fact that it was the software itself, through interaction with the student, forming the environment where learning could be developed.

In recent years an interest on the whole teaching and learning situation has emerged. Progressive consideration has been given to the needs of the teachers who will be using the technology, the ways in which it will be used, the curriculum objectives, the social context and the way in which teaching and learning activities have been planned and organized. Moreover, important consideration has assumed the definition of meaningful practices through which technology can be used effectively. This shift of focus has been particularly evident in all the cases in which technology has been studied in relation to long-term teaching and learning processes of the kind needed for the development of complex articulated knowledge, such as those in which the various TELMA teams are interested in (e.g. arithmetic problem solving, algebraic manipulation, Euclidean geometry, proof in geometry or in algebra, etc.). For the development of such knowledge, the student-software unit of analysis is not sufficing as it is necessary to consider the whole set of interactions established in a class over the course of time and how the activity evolves.

At the theoretical level, we have assisted a progressive move from cognitive theories that emphasize individual thinkers and their isolated minds to theories that emphasize the social nature of cognition and meaning (Resnick, 1987). An increasing importance has been ascribed to theories that highlight the importance of studying the relations among individuals, mediating tools, and the social group.

All teams has approached the design and analysis of experiences involving the use of ICT tools assuming this “olistic” vision of the learning environment, even if the frameworks to which they refer
to in order to define and interpret it are different. For example, some teams (such as, DIDIREM, Did@tic, NKUA) refer to the theory of Didactic Situations for designing, setting up, and analyzing experimentations in which ICT tools are involved, while ITD-CNR and UDE refers to socio-constructivist theories (such as, for example, Activity Theory) for reporting and evaluating projects involving the use of technological tools in school practice.

Let us briefly examine how such approaches have been used to precise the concept of learning environment and how it is possible to (tentatively) compare them.

**How the Theory of didactic situation and Activity theory frame the notion of “learning environment”**

First of all, it is necessary to underline that Activity Theory (AT) conceived to describe different forms of human practices, and not specifically teaching and learning practices in a didactical context. Nevertheless this theory can be used to describe teaching and learning practices. On the contrary, the Theory of Didactic Situations (TDS) considers only teaching and learning practices in the mathematical domain.

Neither the Activity theory, nor the Theory of Didactic Situations are developed to address the learning processes involving interactions with ICT tools. However they have been used within this context by TELMA teams, thus we believe it can be of particular interest to focus on how the these theories can be employed to study teaching and learning process that take place in environments involving ICT tools.

**What terms the two theoretical framework use to indicate the learning environment**

The term “milieu” is used in TDS indicate the environment in which learner construct his/her knowing, that is the learning environment. The milieu is “the set of external conditions within which a human being behaves and grows” (Brousseau, 1988).

The term “milieu” belongs to the Piaget’s genetic constructivism where it is used to describe the concept of adaptation. It is useful to underline that the concept of adaptation is derived from biology and suggests the relationship between living organisms and their environment, with the organisms evolving or fitting in with their environment. The role of milieu in TDS is very important. Brousseau states that that the milieu “plays an important role in the determination of the knowledge that the subject - its antagonist - must develop in order to control a situation for action.”

One of the key concept of Activity Theory is the notion of *activity*. An activity is a form of doing directed to an object, and activities are distinguished from each others according to their objects. The notion of activity is a unit of analysis of the human behavior, an unit of analyze that includes a meaningful context to explain and to interpret individual actions.

Using the framework of the activity theory, we can state that the learning environment is constituted by the enactment of a teaching/learning activity oriented to an educational object, involving student, teachers and artifacts.

Studying the learning environment means studying the teaching/learning activity oriented to a didactical objective.

How do the two theories explain the role of the learning environment when a student acquires new knowledge? In order to answer this question it is necessary to present some key ideas of two theoretical frameworks on which the TDS and the Activity Theory are based, that is the Piaget’s theory and the Vygotskij’s theory. The comparison we are proposing, was partly derived from the works of David Russel and Barbara Rogoff².

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² D. Russell (1994) Piaget’s Structuralism: A reply. This is a part of an interesting discussion between Russel and Brell. URL: [http://jac.gsu.edu/jac/14.1/ReaderResponse/readerresponse5.htm](http://jac.gsu.edu/jac/14.1/ReaderResponse/readerresponse5.htm)

Comparison between the Piaget’s theory and the Vygotskij’s theory

Piaget argues that cognitive development follows stages, that are biologically determined through sensory-motor activity and the universal functioning of equilibration. These stages are universal, and thus common to individuals in all culture, because our bodies have certain specific physiological regularities acting through equilibration and because our experiences with physical objects and forces has certain universal regularities. The categories of thought are also universal, though they manifest themselves in many cultural forms, abstracting from scheme of action, which are rooted in the sensor-motory activities Scheme denotes a property of an action which can be generalized to other content. Piaget saw cognition and behavior arising by means of functions (organization and adaptation) and structural relation. Out of the biological functions of organization and adaptation come cognitive functions (assimilation, accommodation), which in turn give rise to mental categories of reason. Development is fundamentally an individual matter, a substract of structural transformations, a universal cognitive base upon which semiosis and culture rest. Social factors may only influence (foster or impede) children’s natural or inevitable cognitive development. His unit of analysis is the individual.

The development create the condition for the process of learning. Hence formal instruction must wait for students to arrive at an adequate developmental stages to be a positive influence; the asymmetrical power relation of children and adults means that discussion between them are unlikely to result in cognitive restructuring.

Vygotskij has a socio-historical view of development that makes social interaction the center of his theory. Cognition and behavior arise from the interaction of a person with other persons and events in the world, over time, with the use of cultural tools. Humans are biological organism in a physical environment, but very early in ontogeny, social rather than biological factor carry the burden of explanation for cognitive development. The unit of analysis is the cooperative activity, a goal-directed social interaction. The use of activity as unit of analysis – with active and dynamic contribution from individuals, their social partners, and historical tradition and materials and their transformation - allows a reformulation of the relation between the individual and the social and cultural environment in which each is inherently involved in the others’ definition. Children cognitive development had to be understood as taking placed through their interaction with other members of the society who are more conversant with the society’s in intellectual practice and tools for mediating intellectual activity (see the notion of zone of proximal development in Vygotskij).

Vygotskij theory emphasizes the social-historical interaction of students and school, constructing the development in ways unique to each culture. In this view the activity of schooling leads development. The teachers (or more expert peers) are essential partners in development, co-creators, not accompanying facilitators to a process that begins and develops by other means.

The relationship between formal instruction and development can be described through the relationship between spontaneous concepts and scientific concepts.

Comparison between the idea of learning environment in the TDS and in the Activity Theory

TDS: learning environment as milieu antagonist of the subject

The main interest of Piaget’s research was centred on the cognitive development of child. The teaching and learning processes was not a central aspect of his research. On the contrary, the aim of TDS is to study learning in didactical situations in the mathematical domain. Referring to the Piaget’s theory, Brousseau says that student learns by means of adaptations to the milieu which is source of contradictions, difficulties and disequilibria; but a milieu without didactical intentions is not sufficient for the student to acquire the knowledge the teacher would like him/her to acquire.

An important part of the work of devising didactical engineering in Didactical Situations Theory is to find a fundamental situation for teaching a mathematical concept (didactic situation), which will be the point of departure to create an antagonistic system for pupils. Bound by the didactical contract, pupils know they have to behave in a given situation by acting on it. Acting creates retro- actions and
from this dialectical process pupils’ knowledge is born. So teaching, in this theory, needs this antagonistic system which is named the milieu.

In the TDS, learning environment is seen as system that is antagonist to the learner. The milieu opposes retroactions to the answers or to the inadequate choices of the student with respect to the adidactic situation presented. In order to learn, a student has to understand as insufficient his/her control of the situation. The milieu is not an allied of the student but it is a competitor. In fact: “si l'enseignant cherche à organiser un milieu allié où l'acteur agit sous des contraintes qui essayent de lui faire éviter les confrontations, alors nous sommes en face d'interactions de type fictif Dans l'apprentissage par adaptation, il s'agit au contraire de construire des connaissances contre un milieu antagoniste qui résiste. En effet, ce sont les rétroactions du milieu qui permettent l’apprentissage de l’élève. Dans un milieu allié, il n’y a pas de rétroaction, l’élève agit, le milieu " est agit " (Margolinas, 2001)

In order to better describe how the notion of milieu (learning environment) is presented in TDS, it is necessary to put in evidence the role of the teacher and the role of artefacts in teaching learning situations. One of the roles of the teacher is to construct the conditions under which the responsibility of the solution of the task is entirely submitted to the student. This process is named “devolution”. Between the moment in which student accepts the problem as his/her own problem (and not as a school problem) and the moment in which he/she produces the solution, the teacher has to step aside: the student has to construct his/her knowing. Another role of teacher concerns the institutionalisation of the acquired knowledge. The artefacts contribute to structure the material milieu in which student acts. The interaction with artefacts, in fact, gives students retroactions which allow students to develop new strategies of solution.

**AT: Learning environment as cooperative activity oriented to a educational goal**

In the AT frame, the learning environment is constituted by the enactment of a teaching/learning activity oriented to an educational object, involving student, teachers and artifacts. Studying the learning environment means studying the teaching/learning activity oriented to a didactical objective. Moreover, the cooperative and social character of a human activity has been highlighted in the vygotskijan frame. This aspect brings a cooperative connotation to the notion of learning environment. A learning environment is something which is negotiated, co-built in the teaching and learning activity by participant of the activity, and it evolves during the development of the activity. Thus, it is not something which is assigned and constructed a priori.

AT provides a model to describe the structure of any human activity, and its transformations occurring along with its evolution. It is the model proposed by Engestrom and Cole, which can be used also to describe the system of relationships characterizing a teaching/learning activity, and thus to describe a learning environment.

This model assigns a crucial mediation role to the instruments, the rules, and the division of labour in the three relationships characterizing any human activity, that is the relationships between subject and object, between subject and community, between community and object.

By means of this model it is possible to describe the nature of the cooperation that characterizes the activity and that is indispensable for achieving the object.

According to this model, the teacher is a co-actor of the activity, and artefacts are instruments that mediate the subject’s action, the subject-community communication, and the variety of roles, duties and obligations characterising the relationship between community and object.

The model clarify the mediations and the relations that determine the potentialities of fostering learning, highlighting its social nature.

Moreover the model is used to highlight the evolution that an activity can undergo during its development when contradictions or breakdowns occur, forcing a change of focus in the activity, thus forcing a transformation of its structure.
To sum up, the main difference between learning environments in TDS and AT, are that: in the TSD the learning environment is the milieu, that is antagonist to the subject; whilst in AT, the learning environment is the cooperative activity, oriented to an educational aim.


Theory of Didactic Situations: the didactical contract

As previously remarked, the Theory of Didactic situations (TDS) is influenced by Piaget’s theory, but, as stated by Brousseau, in Piaget’s theory the teacher may be discharged of any didactical intentions: a milieu without didactical intentions is not sufficient for the learner acquire all the knowledge the teacher would like him/her to acquire. As a consequence the role of the teacher is thus fundamental in the TDS, so are the relationship between teacher and learner within an ongoing teaching and learning process. Such relationships, in TDS, are called didactical contract.

The didactical contract is not a generic pedagogical contract, because it depends strictly on the corpus of knowledge on which a given teaching and learning process focuses. It is a set of relationships that determines what are the responsibilities of the teacher, and of the learner, with respect to each other. Some of such relationships can be explicitly stated, but most of them are implicit. It is this system of reciprocal obligations that can be defined as being a contract; what characterizes such contract as being didactic is its part specifying educational contents: the aimed mathematical knowledge.

The reciprocal obligations of the contract cannot be enunciated, because of their strongly implicit nature. What results to be particularly important is the breakdown of the contract (ex. A student’s surprise when asked by the teacher to accomplish a task he/she is not able to accomplish; the teacher’s surprise when he/she thought that the student was able to accomplish the task, and that his/her explanation was sufficient, etc.). In fact, also the rules of the contract, like any other form of learning, are interiorized through a process of assimilation and accommodation. Constructing of a didactical contract for the learning of given knowledge takes place through a dynamic process in which contradictions /breakdown may emerge. These appear as breakdowns between what the teacher expects in terms of the student’s acceptance of obligations and the load of responsibility that the student’s is able to bear when tackling tasks.

Overcoming these breakdowns can lead to adaptation phenomena that do not bring about effective knowledge within the class (for example the Topaze effect, the Jourdain effect etc.). On the other hand, the breakdown may be overcome through the search for a new contract based on the readjustment of the previously.

Activity theory: division of labour

As previously stated, the influence of vygotskiyan theory on the Activity Theory determines the cooperative and social character of the activity as the engine of the learning process. Within this context, the relationships between teacher and student can be interpreted as mediators between community and object of the activity. The teacher is part of the community because participates to the activity sharing the same object that the student (subject) has to learn. The relationship between community and object is called division of labor. The division of labor refers to the explicit and implicit organization of community as related to the transformation process of the object into the outcome.

According to the activity theory model, belonging to a community implies a division of labor, that is, the repeated and negotiated distribution of work tasks, power, and responsibilities among the participants. In practice, the division of labor defines a system of reciprocal obligations that mediate the strategy by which community members, interpreting specific roles, interrelate for the social construction of the object of the activity.

The division of labour is built along with the development of the activity; it can evolve by means of breakdowns forcing a change in the focus of the activity. For instance, the teacher may realize that
he/she charged the student of a responsibility that he/she is not able to manage. It may then emerge a contradiction (the student is not able to accomplish a task) that changes the focus of the activity: from “accomplishing the task” to “providing the students with the elements needed to be able to accomplish the task”. This leads the teacher to revise his/her role in the activity assuming, for instance, a more cooperative role and consequently modifying the division of labour.

**Socio-constructivism: socio-mathematical norms**

The notion of socio-mathematical norms can be contextualized in the socio-constructivist framework. The idea is that students construct their knowledge and understanding of the world not just through direct personal experience and discovery (constructivist paradigm), but also through the intellectual sharing and support of those around them (socio-constructivist paradigm). In this perspective the teacher plays a crucial role.

The teacher is perceived in terms of her/his role in organizing the class, setting up tasks for the students to be engaged in and supporting their learning process. Within this context, the relationship between teacher and student, with respect to the learning of a given mathematical concept, is defined in terms of *sociomathematical norms*.

These norms are distinct from general classroom social norms in that they are specific to the mathematical aspects of student’ activity. For example, the understanding that students are expected to explain their solutions and they ways of thinking is a social norms whereas the understanding of what counts as an acceptable mathematical explanation is a socio-mathematical norms.

In general, we may say that the concept of *sociomathematical norms* is comparable with the *didactic contract* of the TDS and with the *division of labour* in the Activity Theory. Indeed they all define the role of the subjects (teacher and students) that are involved in a teaching and learning process with respect to a mathematical concept.

**The role of instruments in teaching and learning processes**

When research on ICT in mathematics education is concerned, one key issue is the concept of instrument, and the role it can play in teaching/learning processes. Such role, in the different researches, depends on the chosen theoretical frameworks, as we are going to highlight in this section. First of all, we clarify what we mean with the word *instrument*. Different theoretical frameworks may ascribe slightly different meanings to the word *instrument*, however, there are some commonalities that we can assume as characterising the concept of instrument. In order to avoid confusion, we analysed several dictionaries of different languages (Italian, English, Spanish, French) and individuated what we will consider (in this section) as being the founding idea of the concept of instrument: *a means whereby something is (or can be) achieved, performed, furthered, or done*. Without going to much into detail on the concept, we can say that whenever we talk about an instrument, we can consider:

- an agent (or subject);
- an instrument (could it be a concrete or non concrete object);
- an objective that the agent tries to achieve by means of the instrument;
- how (in what ways) the given instrument can be a means for achieving the given objective

Individuating such elements can help clarify the role of instruments in teaching/learning processes when ICT in mathematics education is concerned. First of all, we observe, that whenever a teacher (or educator in general) introduces an ICT tool in an educational practice, the tool must be considered as an instrument at two clearly different levels:

1. the level of the *educational instrument*: at this level the *agent* is the teacher, the *objective* corresponds to the specific teacher’s educational goal, and *how* the instrument functions as a means for achieving the *objective* correspond to the modalities of employment of the instrument adopted by the teacher. In other words, this level is defined by the *didactical functionalities* of the ICT tool individuated and exploited by the teacher (RIF PARAGRAFO). We may refer to this level as to the *educational level*.  

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2. the level of the practical instrument: at this level the agent can be any user, and the objective is not an educational goal. We may refer to this level as to the practical level.

For instance, suppose that a teacher’s educational goal is that of introducing pupils to geometrical constructions, he/she may exploit a Dynamic Geometry Software’s didactical functionalities and set up an activity in which pupils are required to construct a square using such an ICT tool. In this case, at the practical level a pupil may use the Dynamic Geometry Software as means for producing the square: the agent is not an educator, and his/her objective is not an educational goal, it is just to produce the square (or maybe other objectives such as getting a good mark). On the other hand, at the educational level, the agent is the teacher, and her/his objective is that of introducing pupils to geometrical constructions, that is an educational goal.

Distinction between the practical and the educational level, will help us in discussing the key idea underlying the researches focused on the use ICT tools for educational purposes. Whatever is the assumed theoretical framework, the key assumption is that whenever an agent employs an instrument as a means for achieving an objective he/she may learn something. Such learning outcome is a result of the use of an instrument at the practical level, and can be exploited by an educator who may use the same instrument at the educational level. Once the instrument is introduced in the educational practice, and the learners use it at the practical level, the resulting learning outcome can be exploited by the educator according to his/her educational goal. However what is learnt, how it is learnt, and how such learning can be fostered, controlled, exploited, is one key issue of the research on ICT tools in mathematics education. Such an issue is approached differently according to the chosen theoretical frameworks, in the following we are sketching some key ideas underlying the frameworks adopted by TELMA teams.

The role of instruments for the theory of didactic situations

When an instrument is employed in school practice, as we stated, we may analyse both, the practical and the educational level.

When a student is using an instrument to solve a problem, we say that he/she using the instrument at the practical level. Within this framework, the learning outcomes resulting from the use of an instrument at the practical level are discussed in terms the interaction of the learner with the milieu antagoniste. If an instrument is a component of the milieu, and if a learner is an agent using the instrument for a practical objective, then we can interpret the interaction between agent and instrument in terms of the interaction of the learner with the milieu antagoniste. If an instrument is a component of the milieu, and if a learner is an agent using the instrument for a practical objective, then we can interpret the interaction between agent and instrument in terms of the interaction of the learner with the milieu; if that is the case, then we may consider the learning outcomes as being the result of the adaptation of the learner to the milieu in consequence to the retroactions of the milieu on the learner himself/herself. Thus, if an educator wants to employ an instrument at the educational level he/she has to set up situations in which the instrument is part of the milieu and is employed by the learner as a mean to accomplish the proposed task. The nature of the learning outcomes is discussed in terms of situation that is being set up, and in terms of the nature of the milieu, thus it is particularly important to study the nature of the employed instrument (ICT tool in our case), and its retroactions on the user, according to the chosen educational goal.

The role of instruments for the activity theory

In the case of the Activity Theory, an instrument is considered to be mediating both the relationship between subject and objective of an activity, and the relationship between subject and community in the sense that the introduction of an instrument may change the rules of the community and the division of labour. Within this theory, the learning environment is not considered as antagonist to the subjects (as in the case of the milieu antagoniste of the didactic situations theory), on the contrary, it is considered to be a cooperative environment. The interaction between a learner and the learning environment is interpreted in terms of cooperation, and not in terms of opposition, in this sense, the agent using an instrument to achieve an objective, is considered to be cooperating with the instrument as being part of the learning environment. When a learner uses an instrument at the practical level for achieving an objective within an activity, the learning outcomes are considered to
be structured by the nature of the activity itself and by role played by all its components. Consequently, within this theory, in order to employ an instrument as an educational instrument for achieving a given educational goal, an educator has to set up an activity in which the instrument mediates the relationship between the learner and a objective (at the practical level) that is relevant for the given educational goal. Moreover, for the activity to be effective with respect to the chosen educational goal, the educator has to consider how the employed instrument structures the activity and the relationships among the different components of the activity itself. In other words, the educational functionalities of the instrument are defined in terms of how the instrument may structure an activity, rather then on the retroactions given to the user as in the case of didactic situation theory; of course also such retroactions are to be considered, because they influence the relation between learner and instrument, but they are not they main focus.

The role of instruments for the Rabardel’s theory

According to this theory, when an subject uses an instrument to accomplish a given kind of task, he/she passes through the process of instrumentation that results in the internalisation of the schemes of use (Rabardel, 1995) of the instrument in the form of techniques that can be applied to a whole class of tasks that are somehow compatible with the given kind of task. In other words, when an instrument is used at the practical level, the process of instrumentation results in a learning outcome consisting on the internalization of certain techniques associated to the schemes of use of the instrument, and that can be applied to a wide class of tasks. Suppose that an educator aims at a given mathematical educational goal which in particular involves pupils to learn how to accomplish certain tasks, or to learn certain mathematical techniques, then it is possible to employ a suitable instrument as an educational instrument. The educator may chose an instrument to be used by the learners in order to accomplish certain tasks that are relevant to the given educational goal. Then the instrumentation process may lead the learner to internalize the schemes of use of the instrument in the form of techniques that are coherent to the aimed mathematical ones, and that can be applied to the mathematical tasks individuate by the educational aim. Within this framework, the didactical functionalities of an ICT tool are strongly dependant on its schemes of use that structure the learning outcomes derived from the employment of the instrument at the practical level. As a consequence, in order to employ an instrument as an educational instrument a special attention has to be put on the schemes of use of the instrument when it is used to accomplish a given kind of tasks

Annexes : The theoretical frameworks of reference

In this section we present the theoretical frameworks of reference used by TELMA teams. We have classified these frameworks considering their nature (theoretical framework or theoretical construct) and their origins (Educational theories, Artificial intelligence theory and so on). In particular, we have individuated the following:

- Educational theories (The anthropological theory, The theory of didactic situations, Microworlds related theories, Theory of instrument of semiotic mediation)
- Educational theoretical constructs (Socialmathematical norms, Model cKé of conception)
- Theories used with educational aim (Rabardel’s theory, Linguistic theories, Activity theory)
- Artificial intelligence theories

For each theory we present an overview, a short description of the theory and, when it makes sense, we illustrate the possible role of ICT tool in the theory.
Educational theories

The anthropological theory

Overview

The anthropological theory is a didactical theory developed by Chevallard (1992). The anthropological approach shares with “socio-cultural” approaches in the educational field (Sierpinska and Lerman, 1996) the vision that mathematics is seen as the product of a human activity.

Description of the theory

“Mathematical objects are not absolute objects, but are entities which arise from the practices of given institutions. To analyse a mathematical object in an institution, to understand the meaning in the institution of “knowing/understanding this object”, one needs to identify and analyse the practices which bring it into play. These practices, or “praxeologies”, as they are called in Chevallard’s approach, are described by four components: a type of task in which the object is embedded; the techniques used to solve this type of task; the “technology”, that is to say the discourse which is used in order to both explain and justify these techniques; and the “theory” which provides a structural basis for the technological discourse itself and can be seen as a technology of the technology. In this way the technological and theoretical components are combined into a single “theoretical” component. A technique is a manner of solving a task and, as soon as one goes beyond the body of routine tasks for a given institution, each technique is a complex assembly of reasoning and routine work. Techniques are most often perceived and evaluated in terms of pragmatic value, that is to say, by focusing on their productive potential (efficiency, cost, field of validity). But they have also an epistemic value, as they contribute to the understanding of the objects they involve, and thus techniques are a source of questions about mathematical knowledge.

The advance of knowledge in any institution requires the routinisation of some techniques. A technique which has become routine in an institution tends thus to become “de-mathematicised” for the members of that institution. It is important to be aware of this naturalisation process, because through this process techniques lose their mathematical “nobility” and become simple acts.”

(Artigue M.,2002)

The role of ICT tools

The main focus of the anthropological theory, is not on ICT tools, however they may be addressed as shown by Artigue:

“The great reduction in the cost of execution that technology offers, reduces the need for routinisation work. Techniques that are instrumented by computer technology are changed, and this changes both their pragmatic and epistemic values. The mathematical needs of the techniques change also: new needs emerge, linked to the computer implementation of mathematical knowledge and the representation system involved. The anthropological approach furnishes an effective framework for questioning these changes and their possible effects on mathematics teaching and learning”

(Artigue M.,2002)

The theory of didactic situations

Overview

The Theory of Didactic Situations is developed in the domain of mathematics education by Guy Brousseau in 1997, and it is founded on Piaget’s psychogenetic theory.

According to this theory, the subject learns by adaptations to the “milieu” which produces contradictions, difficulties and disequilibria. Thus, the role of the teacher is that of setting up the milieu with according to a didactical aim: he/she plans an adidactical situation, that is a situation ad hoc in which learners may achieve the wished adaptations.

Description of the theory

The main aspects of such theory, as described and commented by Anna Sierpinska (http://alcor.concordia.ca/~sierp/TDS.html), are briefly presented in the following.

“What is a didactic situation?
The notion of didactic situation is grounded in certain assumptions about learning and teaching. These assumptions can be formulated using the metaphor of game (…).

* the teacher is a player faced with a system composed of a student and a didactic milieu
* the student is himself a player in a game of him/herself with a didactic milieu
* in the student’s game with the didactic milieu, knowledge is the means of understanding the ground rules and strategies and later, the means of elaborating winning strategies
* the teacher’s aim is to engage the student in such a game; aiming at a particular mathematical knowledge, the teacher will try to set the student-milieu system so that, indeed, this knowledge would appear as the best means available for the understanding of the rules of the game and elaborating the winning strategy”

The teacher sets up situations in which learners can construct their own relationship with knowings as a response to the exigencies of the milieu rather than as result to the teacher expectations. This kind of situation is defined by Brousseau as “adidactical situation”. In an adidactical situation there is a shift of responsibility with respect to the knowings from the teacher to the student: in an adidactical situation the student becomes directly responsible of his/her own relationship with knowings.

[...]We have assumed that the didactic situation can be described as a game between a (person in the role or position of the) teacher and the student-milieu system. Every game has its rules and strategies. The rules and strategies of the game between the teacher and the student-milieu system, which are specific of the knowledge taught, are called the ‘didactic contract’3. (p. 41). The assumption that the rules taken into account are pertinent from the point of view of the knowledge taught or aimed at is essential for the definition of the didactic contract. If the rules taken into account have nothing or little to do with this knowledge, but are relevant from the point of view of, say, classroom management, or political correctness, or the general culture, then we might speak of other types of contract, maybe (e.g. pedagogical contract) but not of a didactic contract.

(A. Sierpinska, Lecture 3, 1999)

The teaching situation can be described as a set of relations between the teacher and the system student-milieu, for this reason it seems necessary to discuss more in details the notion of milieu. Such a notion is described by Serpinska as follows:

“WHAT IS A MILIEU?
‘Milieu’ should perhaps be understood in an ecological sense, as in ‘water is the natural milieu of fish’. Thus the ‘didactic milieu is the natural milieu of students’. A person, a human being, normally lives in several different milieus and plays different roles in them. In a family milieu one can be a child, a mother, a father, etc. In a sports milieu, one can be the player, the coach, etc. Other possible roles could be played in a workplace milieu, social milieu, etc. In the school milieu, one can be a student, a teacher or an administrator. In each course, the student has to cope with a specific milieu, and there are even more specific milieus for each class in a course. To ‘survive’ (‘to win’) in a milieu one has to get to know the ‘rules of the game’ and develop strategies for winning the game.

DIFFERENCE BETWEEN THIS APPROACH TO TEACHING AND LEARNING AND THE TRADITIONAL POINT OF VIEW
Learning is not reduced to the result of a transmission of information from teacher to students. Learning is understood more as sense making of situations in a milieu, and developing ways of coping with them. Teaching of a knowledge K consists in organizing the didactic milieu in such a way that knowledge K becomes necessary for the student to survive in it. If the situations in a mathematics class are such that a certain type of social behavior is sufficient for survival in them, without any use of mathematical knowledge, then it is the social behavior, not the mathematical knowledge that the students will learn. If the teacher solves the problems for the students and only asks them to reproduce the solutions, they will learn how to reproduce teacher’s solutions, not how to solve problems. In this sense, the kind of game the student has to play with the milieu, to survive in it, determines the kind of knowledge that he or she will acquire. Thus, in the theory of situations, ‘knowledge is [understood as] the outcome of the interactions between the student and a specific milieu organized by the teacher in the framework of a didactic situation (Balacheff, 1993).”

3 Siamo noi che sottolineiamo questo termine.
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ERT TELMA

In order to better characterize the idea of milieu we report an excerpt of a paper by Balacheff and Gaudin:

“We will call milieu such a subset of the environment of a subject; the milieu is a kind of projection of the environment onto its epistemic dimension. Indeed, in the case of mathematics, knowings are not only the consequences of the interaction between a subject and a material milieu, but they involve also interactions with systems of signifiers produced by the subject herself, or by others. We must then extend the classical idea of milieu in order to integrate symbolic systems and social interaction as means for the production of knowings. This is the meaning of Brousseau’s proposal to define the milieu as the subject's antagonist system in the learning process (Brousseau, 1997). So, we do not consider a knowing as a property which can be ascribed only to the subject, nor only to the milieu. On the contrary we consider it as a property of the interaction between the subject and the milieu—its antagonist system. This interaction is meaningful because it succeeds in fulfilling the necessary conditions for the viability of the subject/milieu system. By viability we mean that the subject/milieu system has a capacity to recover an equilibrium following some perturbations; what implies that the perturbation is recognized by the subject (for example, a contradiction or an uncertainty). In some cases the subject/milieu system may even evolve if the perturbations are such that this is necessary.”

(Balacheff, N., Gaudin, N., 2002).

The role of ICT

In the theory of didactic situations the role of ICT tool is not directly addressed, however, according to the definition of milieu, we may consider ICT tools as elements of the milieu that a teacher provides to his/her pupils when a didactic situation is set up.

Microworlds related theories

Overview

Starting from Papert’s idea of Microworld as environment embedding a knowledge domain, several theories aiming at exploiting this idea has been developed, here we give a brief overview of their key ideas.

Description of the theory

According to Bottino (Bottino, 2001) the meaning of the word microworld is not a standard among researchers, but there is a certain level of agreement on what characterises such kind of systems. In particular the key idea of the concept of microworld is that of an environment, characterised by a knowledge domain which thus becomes explorable, as suggested by the following characterisation, given by Balacheff and Kaputt:

“A microworld consists of the following interrelated essential features:

i) a set of primitive objects, elementary operations on these objects, and rules expressing the ways the operations can be performed and associated - which is the usual structure of a formal system in the mathematical sense.

ii) a domain of phenomenology that relates objects and actions on the underlying objects to phenomena at the 'surface of the screen'. This domain of phenomenology determines the type of feedback the microworld produces as a consequence of user actions and decisions (Balacheff & Sutherland, 1994)”

(Balacheff & Kaput, 1996).

At the core of the relationship between the user and the knowledge domain there are the objects of the interface that are available to the user. Papert termed them transitional objects "standing between the concrete/manipulable and the formal/abstract" (as cited by Noss and Hoyles, 1996). Such objects are the means of the interaction between the user and the environment, thus, between the learner and the knowledge domain.

A microworld is thus an environment where exploration is possible thanks to transitional objects, but where such exploration is constrained in ways designed to promote learning; knowledge is assumed to be reflected in the system, in its elements, relationships and structures, and it is assumed to be evolving by means of actions within the system. In particular, a microworld has to be able to reflect evolvable knowledge along the course of activities, thus it has to be evolvable itself.
To sum up three main characteristics of microworlds are: that they embed a knowledge domain model; that they offer transitional objects to the user to act with; that they can be extended.

The focus on the nature of computational objects as the central elements in a microworld, the choice of which is critical:

“The choice of such objects and the ways in which relationships between them are represented within the medium, are critical. Each object is a conceptual building block instantiated on the screen, which students may construct and reconstruct.”

(Noss and Hoyles, 1996).

The effectiveness of computational objects depends on their capability to stimulate learners and on their capability to foster intuitions, understandings and personal images, together with preferences and pleasure. By means of computational objects, microworlds can be used to fill the gap from users' existing experiences and static formal systems, a gap which is often too great for learners to engage directly with such systems. On the other hand, the direct interaction with the computational objects offers a chance of connecting learners' knowledge with the knowledge domain represented by the microworld. In particular, mathematical microworlds should forge links with mathematical objects and relationships, this aspect distinguishes them from a simply playful exploratory world which is mathematically uninteresting.

Among microworlds related theories we find the theoretical construct of “situated abstraction” developed by Noss and Hoyles (Noss and Hoyles, 1996).

According this theoretical tool, abstraction does not come ready-made. It is a process which develops in activity, which – like all activity – is situated. From this perspective, the process of abstraction can be seen as a way of layering meanings on each other, rather than as a way of replacing one kind of meaning (concrete, referential) with another (abstract, decontextualised). The hypothesis is that this process can happen especially felicitously in computational worlds. In such settings, what can and does happen, is that meanings become reshaped as learners move the focus of their attention onto new objects and relationships within the setting, while maintaining their connections with existing ones (Noss, Healy and Hoyles, 1997).

Theory of instruments of semiotic mediation

Overview

This theory has been developed on the basis of vigotskjian theories; it addresses the problem of how it is possible to exploit artefacts for educational aims, focusing on the semiotic processes that can be derived from the introduction of artefacts in school practice.

Description of the theory

For what concerns the use of artifacts in general, thus also of ICT tools, this framework is based on the following key hypothesis:

1. on one hand meanings are rooted in the phenomenological experience
2. on the other hand such meanings can evolve by means of social constructions in the classroom under the guidance of the teacher.

In particular learning outcomes resulting from an activity with ICT tools are assumed not to be necessarily consistent with the educational aims individuated by the teacher. However, an evolution of what pupils learn toward coherency with mathematics is needed, such evolution can be achieved through social construction under the guidance of the teacher.

In order to better clarify the key ideas of the framework we shall consider the distinction between instrument and instrument of semiotic mediation, proposed by Mariotti:

“[…] as far as the computer is concerned, it intervenes in the activity, but it participates to it in different ways, according to the actors involved in that activity:
- as an instrument (the artefact is used according to utilisation schemes); and in that case meanings, if any, may emerge, but the mathematical meaning, embedded in it, may remain inaccessible to the user.
- as an instrument of semiotic mediation; as the teacher utilises it in order to accomplish communication strategies aimed to develop a specific meaning, related to the mathematics content which constitute the motive of the teaching/learning activity.”

(Mariotti, 2002)

The key element of this theory is thus the idea of instrument of semiotic mediation, which refers to a special use of instruments in class practices: the instrument is introduced in the practices on purpose by the teachers, and it is exploited to accomplish communication strategies that aim at developing meanings related to the mathematical contents consistent with the motive of the teaching/learning activity.

The evolution of meanings, within this framework, is based on the idea that of deriving, from a used instrument, hybrid signs which refer both to the practice with the instrument, and to the sphere of theory of the mathematical knowledge corresponding to the teacher’s educational objective. Such hybrid signs can be used as pivots for directing the focus of activities and discourses either toward the sphere of practice or toward the sphere of theory. Movements back and forth the two spheres are exploited to convey practical meanings to the sphere of theory and theoretical meanings to the sphere of practices, allowing to exploit microworlds as means for generating theoretical meanings. In this framework, how theoretical meanings are originated from phenomenological experience depends strictly on how the teacher exploits hybrid signs as pivot, structuring a complex relationship between the considered microworld or artefact, and the mathematical knowledge corresponding to her educational objective. Meanings are developed under the guidance, thus under the control, of the teacher who is institutionally in charge of ensuring their consistency with mathematical knowledge.

This theory is used by the Pisa team both at the level of realizing ICT tools, and employing them in the educational experiment cycle. In fact the framework has been used in order to individuate and exploit the didactical functionalities of the ICT Cabri. Furthermore, it has been used in the planning, put in practice and diagnostic phases of an educational experiment cycle aiming at introducing pupils to geometry theory. In particular the framework has been used to individuate special communication strategies to be employed by the teacher. At the same time the framework has been used to realize and experiment the ICT tool L’Algebrista. In particular it has been used to define the didactical functionalities of the ICT tool, in terms of the educational goal of introducing pupils to algebra as a theory. The tool has been designed in order to favor the employment of particular teacher’s communication strategies that thus influenced the definition of the tool’s didactical functionalities, which in this framework depend strictly on the role played by the teacher.

Educational Theoretical constructs

Socialmathematical norms (Cobb & Yackel, 1996)

Overview

This theoretical construct, belonging to the socio-constructivist paradigm, allows a way of interpreting mathematics classrooms that aims to account for how students develop mathematical beliefs and values and, consequently, how they become intellectually autonomous in mathematics.

Description of the theoretical construct

Before giving a definition about socialmathematical norms we try to explain what it means “norm” and in particular “classroom norm”.

“"Norm" is a sociological construct and refers to understandings or interpretations that become normative or taken-as-shared by the group. [...] One way to describe classroom norms, is to describe the expectations and obligations that become "normative." In saying this, I mean to imply
that these are the expectations and obligations that become constituted in the classroom. Classroom interactions can be characterized in terms of these expectations and obligations”. (Yackel, 2000)

Socialmathematical norms are classroom norms, but they are distinct from general classroom social norms in that they are specific to the mathematical aspects of students’ activity.

Social norms are often introduced as expectations of the teacher relating to students' actions, but sociomathematical norms must grow out of the students' involvement with the teacher, each other, and appropriate mathematical activities. Sociomathematical norms are involved with deciding what is appropriate, sufficient, and different in mathematical discussions, procedures and solutions.

For example, normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient and mathematically elegant in a classroom are socialmathematical norms. Similarly what counts as an acceptable mathematical explanation and justification is a socialmathematical norms whereas the understandings that students are expected to explain their solutions and their ways of thinking is a social norm.

Socialmathematical norms are not predetermined criteria introduced into the classroom from the outside. They are continually regenerated and modified by the students and the teacher through their interactions. Teachers may have clear ideas in advance of norms that they might wish to foster; but even in such cases these norms are interactively constituted by each classroom community.

On the other hand, in the process of negotiating, social and socio-mathematical norms, students construct and reconstruct personal beliefs and values that help them become increasingly autonomous in mathematics. Social norms benefit students by developing social autonomy, while sociomathematical norms benefit students by developing intellectual autonomy (Yackel & Cobb, 1996). These sociomathematical norms become an intrinsic aspect of the mathematical culture of the classroom and serve to heighten the intellectual autonomy of the students. In this sense, socialmathematical norms and goals and beliefs about mathematical activity and learning are related. “What becomes mathematically normative in a classroom is constrained by the current goals, beliefs, suppositions, assumptions of the classroom participants. At the same time, these goals and largely implicit understandings are themselves influenced by what is legitimized as acceptable mathematical activity” (Yackel, Cobb, 1996).

Conceptions theory (Model cKë)

Overview

The word “conception” has been used for years in research on teaching and learning mathematics, but it has been used as a common sense notion. The model cKë is developed to answer to the need for a better-grounded definition of conceptions, and for tools allowing to analyze their differences and commonalities. This model gives a form of formalization of the notion of “conception”.

Description of the theoretical construct

“We call conception C a quadruplet (P, R, L, Σ) in which:

- P is a set of problems;
- R is a set of operators;
- L is a representation system;
- Σ is a control structure.

The informed reader will recognize, underlying the three first components, the key features identified by Vergnaud (Vergnaud, 1991) in order to characterize a concept; we have introduced the fourth one for reasons we explain hereafter. The very first question of any researcher in mathematics education will be that of knowing how to relate this formal definition with the “reality” he or she is faced to. We will consider this point for each of the four elements of the definition.

The question of the concrete characterization of the set P of problems, is complex. […] we propose to adopt a pragmatic position, deriving the description of P, in an empirical way, from the characterization of situations allowing to diagnose students’ conceptions. This approach can be strengthened by the analysis of historical and actual uses of mathematics (e.g. Sierpinska 1989, Thurston 1994, d’Ambrosio 1993, Lave 1988, Nuñes et al. 1983).
The question of the concrete characterization of the set $R$ of operators is more classical. Operators are means to obtain an evolution of the relations between the subject and the milieu; they are the tools for action. Operators could be “concrete”, allowing to perform actions on a material milieu, or “abstract”, allowing to transform linguistic, or symbolic, or graphical representations. So, an operator could take the form of functionality at the interface of a software or of a syntactic rule to transform an algebra expression, or it could even take the form of a theorem in an inference.

The representation system $L$ consists of a repertory of structured set of signifiers, of a linguistic nature or not, used at the interface between the subject and the milieu, supporting action and feedback, operations and decisions. Just to mention few examples: algebraic language, geometrical drawing, natural language, but also interfaces of mathematical software and calculators are all examples of representation systems. Whatever it is, depending on the state of the subject/milieu system, the representation system must be adequate to give account of the problems and to allow performing operators.

The last dimension of a conception, the control structure $\Sigma$, is constituted by all the means needed in order to make choices, to take decisions, as well as to express judgment. This dimension is often left implicit although one may realize that the criteria which allow to decide whether an action is relevant or not, or that a problem is solved, is a crucial element of the understanding of a mathematical concept. We would suggest that in the Vergnaud proposition the control structure is implied by his reference to theorems-in-action or to inference (Vergnaud 1991), which are meaningful notions only to the extent that they are associated with the recognition that the subject has procedures to check that her actions are legitimate and correct […]

It is important to insist on the fact that this characterization of a conception is not more related to the subject than to the milieu with which he or she interacts. On the contrary, it allows a characterization of the subject/milieu system: the representation system allows the formulation and the use of the operators by the active sender (the subject) as well as the reactive receiver (the milieu). The control structure allows to express the means of the subject to decide of the adequacy and validity of an action, as well as the criteria of the milieu for selecting a feedback.”

(Balacheff, N., Gaudin, N., 2002).

**Relationship between theoretical construct and ICT tool**

The model cK¢ can be used in order to develop an ICT tool to which allows to identify some of the students conceptions mobilized during a resolution problem. It is in fact possible to build systems that individuates (on the basis of student’s actions) the operators and control structures employed by the student within a given problem solving context; the set of such elements constitutes a set of possible conceptions mobilized during the resolution of the problem. In this sense this theory is employed by the Did@TIC team mainly in the phase of design and realization of the software, but also in the diagnostic phase of the educational experiment cycle.

**Theories used in the educational domain**

**Rabardel’s theory**

**Overview**

A theory which is quite popular in recent researches on ICT is the theory described, in the domain of Ergonomy, by Rabardel (Rabardel 1995). This theory is used as support to the educational research of mathematics too.

**Description of the theory**

The key ideas of the theory are described by Artigue as follows:

“The instrument is differentiated from the object, material or symbolic, on which it is based and for which is used the term “artefact”. Thus an instrument is a mixed entity, part artefact, part cognitive schemes which make it an instrument. For a given individual, the artefact becomes an instrument through a process, called instrumental genesis, involving the construction of personal schemes or, more generally, the appropriation of social pre-existing schemes. Instrumental genesis works in two directions. Firstly, it is directed towards the artefact, loading it progressively with potentialities, and eventually transforming it for specific uses; this is called the instrumentalisation of the artefact. Secondly, instrumental genesis is directed towards the subject, leading to the development or appropriation of schemes of instrumented action which progressively take shape as techniques that permit an effective response to given tasks. The latter direction is properly called instrumentation.”

(Artigue 2002)
Linguistic theories

Overview
The Social Semiotic theory and in particular the work of M. Halliday are linguistic theories which emphasize the ways in which language functions in the representations of our experiences and of our social relationship. These theories are not conceived in the didactical domain of mathematics but, in recent years, mathematics education research puts attention to social and linguistic context in order to study the role of language as the principal medium in which teaching and learning take place.

Description of the theory
An innovative point of view to investigate teaching and learning mathematics practices is given by Halliday’s linguistic theory and by the Social Semiotic theory. We present some of their main aspects:

Systemic-Functional Linguistics (SFL) is a theory of language centred around the notion of language function. While SFL accounts for the syntactic structure of language, it places the function of language as central (what language does, and how it does it), in preference to more structural approaches, which place the elements of language and their combinations as central. SFL starts at social context, and looks at how language both acts upon, and is constrained by, this social context.

Language as Social Semiotic consists of a series of essays that extend Saussure's observation that “Language is a social fact”. For Halliday, “Language as social semiotic” means “interpreting language within a sociocultural context, in which the culture itself is interpreted in semiotic terms as an information system” (Halliday, 1993). Language consists of exchanges of meaning in various interpersonal contexts. Language, as Halliday explains, “does not consist of sentences; it consists of text, or discourse”. People in their everyday linguistic exchanges “act out the social structure, affirming their own statuses and roles, establishing and transmitting the shared systems of value and knowledge”. Linguistic research has demonstrated how variation in the linguistic system expresses variation in social status and roles. Halliday specifies these variations in language use in his discussion of dialect and register:

“...dialect variation expresses the diversity of social structures (social hierarchies of all kinds), while register variation expresses the diversity of social processes -- what we do is affected by who we are...the division of labour is social....”

(Halliday, 1993)

Language both expresses and actively symbolizes social structures and systems. This “twofold function” of language empowers modes of meaning as diverse as “backyard gossip to narrative fiction and epic poetry”.

In the collection of essays that constitutes Language as Social Semiotic, Halliday views language from the outside rather than as an “elegant self-contained system”. Halliday's orientation departs from the view of language as formal logical relations. In a linguistics focused on functional exchange of meaning “the conceptual framework is likely to be drawn from rhetoric rather than from logic, and the grammar is likely to be a grammar of choices, rather than of rules” (Hallyday, 1993)

Activity theory

Overview
Activity theory can be considered as a means of structuring and guiding field studies of human-computer interaction, from practical design to theoretical development. It is a psychological theory with a naturalistic emphasis that came to fruition in the Soviet Union in the work of Lev Vygotsky. It provides a hierarchical framework for describing activity and offers a set of
perspectives on practice. It has been applied in many areas of human need, including educational studies.

**Description of the theory**

Some aspects of the activity theory is described as follows by Bottino and Chiappini:

“Activity theory is a philosophical and cross disciplinary theory for studying different forms of human practices, such as teaching-learning practice, as development processes mediated by artefacts, in which individual and social levels are interlinked at the same time (Kuutti, 1996). […] In Activity theory, an activity is a form of acting directed toward an object, and it is the object that distinguishes one activity from another. Transforming the object into an outcome motivates the existence of an activity. Activities consist of actions or chains of actions, which in turn consist of operations. […] Cole and Engeström(1991) devised a model to formulate the complex relationships between elements in an activity [see Fig.2] that is particularly appropriate to study the relationships that take place in the teaching-learning activity (see also Engeström, 1987,1991). Their systemic model highlights three mutual relationships involved in every activity, namely, the relationship between subject and object, that between subject and community, and that between community and object. Each of these relationships is mediated by a third entity. The relationship between subject and object is mediated by tools (i.e. an ICT-based tool) that both support and constrain the subject’s action; the relationship between subject and community is mediated by rules (explicit or implicit norms, conventions, and social interactions) whereas that between community and object is mediated by the division of labour (different roles characterizing labour organization). The model also points out that each entity mediates all the relationships described in the model. For example, the tools used in the activity mediate not only the relationship between the subject and the object but also that between subject and community and that between community and object. Moreover, mediating entities are not mutually independent but exert influence on one another. For example, the introduction of a new tool in an activity influences both the norms regulating participant interaction in the activity and the roles that the participants can assume.”

(Bottino R. M. and Chiappini G., 2002)

![Figura 2: Cole and Engeström's model](image)

**Role of ICT**

In the activity theory the nature of any artifact can be understood only within the context of human activity, by identifying the ways people use this artifact, the needs it serves, and history of its development.

Activity theory can be used to analyse both the use of ICT tools involved in a situation and the design of ICT tools.

Human usually uses an ICT tool because he wants to reach a goal. But the use of a particular tool can change the structure of activity and can result in new goals to be satisfied. Activity theory allows to analyse this change.
For what concerning the design of ICT tools, “Activity theory can make an important impact on the
development of design support tools. The design of a new ICT tool involves the design of a new
activity. However, even the perfect design of an ideal activity does not guarantee the success of a
system. The transformation of an activity from an initial target state can be difficult and even
painful. Activity theory can be used to develop a representational framework that will help
designers to capture current practice and build predictive models of activity dynamics. Such
conceptual tools would enable designers to achieve appropriate design solutions, especially during
the early phases of design.” (Kaptelinin, V. 1997)

Artificial Intelligence theories

Overview

First of all, it is necessary to underline that Artificial Intelligence theories are mainly used for
developing ICT tools and they are not conceived specifically to describe teaching and learning
practices in a didactical context.

We present two of these theories concerning algebra: the Rewriting Rules theory and the Resolution
Theory.

Description of the theory

The Rewriting Rules Theory is described by Nicaud as follows:

“The rewriting rule theory [Dershowitz & Jouanna 1989] provides a computational framework with syntactic
expressions (also called terms) and rewriting rules for transforming expressions according to the replacement of
equals inference mechanism. Important concepts are defined in this theory, in particular: (1) the termination of a
set R of rewriting rules (R is terminating if no infinite succession of rewritings of an expression can be generated
with R), (2) the confluence of a set R of rewriting rules (R is confluent if for any expressions a and b rewritten
from an expression e with R, there exists an expression c in which a and b can be rewritten with R) and (3)
normal forms for a terminating and confluent set R of rewriting rules (the normal form of a form a is the unique
form b in which a is rewritten by applying the rules of R until termination).

Rewriting rules are particularly suitable for modeling, and implementing on a computer, a solver when, in the
problem domain, there exists a set of rewriting rules that terminates on solved forms. But the rules used by
humans in algebra are not terminating: For factoring polynomials, factoring rules are used but sometimes
development rules are necessary, and these rules are inverse rules so that they allow infinite rewritings. In
addition to this problem, humans uses term concepts (e.g., monomial, square) for solving algebraic problems.
These concepts are not currently modeled in the rewriting rule theory.”

(Nicaud, 1994).

The Resolution Theory developed in the polynomial factoring domain is proposed by Nicaud as
follows:

“At strategic level, the resolution theory we have elaborated for the polynomial factoring task includes term
concepts, transformation rules and basic transformation concepts that have been defined from an analysis of
human behavior.

Furthermore, we have defined three concepts of degree: (1) the formal degree is a degree defined in an
expression by analogy with the degree of a polynomial, but it depends only on the expression regardless of the
polynomial (e.g., for 6x^2-6x^2-2x+8 it is 2, although the degree of the corresponding polynomial is 1); (2) the
factorization degree is the number of factors having a formal degree different from 0 (e.g., for (x^3-9)(x-2)^2 it is
3); (3) the additive multi-sets of formal degrees is mainly used for defining a sort of reduction, for a sum it is the
multi-set of formal degrees of each additive expression (e.g., for 4x^3+(x-2)(5x+1)+7x^2, it is \{3, 2, 2\}).

Then we have defined transformation concepts with these degrees. We call a factorization with a strong
significance any transformation u \not\rightarrow v such as u is a sum, the formal degree does not change and the
factorization degree increases. We call a development with a strong significance any transformation u \not\rightarrow v such
as v is a sum, the formal degree does not change and the factorization degree decreases. We call a grouping any
transformation u \not\rightarrow v such as the formal degree does not change and the additive multi-set of formal degrees
decreases. We call a collapsing any transformation u \not\rightarrow v such as the formal degree decreases. We call a
reduction with a strong significance any grouping or collapsing.

With these concepts, we proved the following theorem:
The factorizations and reductions with a strong significance terminate within the set of expressions.

Considering this theorem and considering the fact that in solution paths of polynomial factorings, there is a very high rate of application of factorizations and reductions with a strong significance, we can stand the following strategic principle:

For factorizing polynomials, use mainly factorizations and reductions with a strong significance.

The above principle with a few heuristic rules looking at context of the sub-expression to be transformed and a few heuristic rules looking ahead (rules corresponding to mental calculations) allow to build adequate reference solvers for a large class of problem. This is also a basis for performing the other functions of an ITS (explanation, analysis and help), however other elements and mechanisms have to be added for that purpose.”

(Nicaud. 1994)

Role of ICT

Artificial intelligence theories are used to realize ICT tools. For example, the theories previous presented are employed by the Did@TIC team for the realization of the ICT tool Aplusix.

References


